Closed Bounded Subsets of $\mathbb{E}^n$

Definition A set $C \subset \mathbb{E}^n$ is called a bounded subset of $\mathbb{E}^n$ if there exists a ball $B_r(p) = \{x \in \mathbb{E}^n \mid |x-p| < r\}$ such that $C \subset B_r(p)$.

Definition Let X be a set topological space and let $\mathscr{F} = \{U \mid U \subseteq X\}$ be a collection of subsets of X. Then $\mathscr{F}$ is called a cover of X if The union of the elements of $\mathscr{F}$ is all of X, i.e.

$$\bigcup_{U \in \mathscr{F}} U = X.$$

$\mathscr{F}$ is called an open cover of X if

(1) every $U \in \mathscr{F}$ is an open subset of X.

(2) The union of the elements of $\mathscr{F}$ is all of X, i.e.

$$\bigcup_{U \in \mathscr{F}} U = X.$$

$\mathscr{F}'$ is called a subcover of $\mathscr{F}$ if

(1) $\mathscr{F}' \subseteq \mathscr{F}$

(2) The union of the elements of $\mathscr{F}'$ is all of X, i.e.

$$\bigcup_{U \in \mathscr{F}'} U = X.$$

$\mathscr{F}'$ is called a finite subcover of $\mathscr{F}$ if

(1) $\mathscr{F}'$ is a subcover of $\mathscr{F}$

(2) $\mathscr{F}'$ contains finite number of elements of $\mathscr{F}$.

Theorem A subset X of $\mathbb{E}^n$ is closed and bounded if and only if every open cover $\mathscr{F}$ of X (with the induced topology) has a finite subcover.

Motivated by this result we make the following definition.

Definition A topological space X is compact if every open cover of X has a finite subcover.

Remark With this terminology, the preceding Theorem can be restated as follows.

The closed bounded subsets of a Euclidean space are precisely those subsets which (when given the induced topology) are compact.

Properties of Compact Spaces

(Heine-Borel Theorem) A closed interval $[a, b]$ of the real line $\mathbb{R}$ is compact.

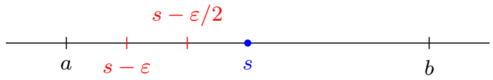
Proof Let $\mathscr{F}$ be an open cover of $[a, b]$. Define a subset X of $[a, b]$ by

$$X = \{x \in [a, b] \mid [a, x] \text{ is contained in the union of a finite subfamily of } \mathscr{F}\}.$$

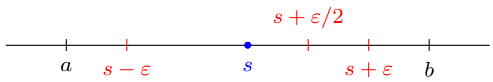
Since $a \in X$ and $X \subseteq [a, b]$, $X \neq \emptyset$ and is bounded above by b, so $s = \sup X$ exists.

Claim

- $s \in X$, i.e $[a, s]$ is contained in the union of a finite subfamily of $\mathscr{F}$.
  Proof Let $O \in \mathscr{F}$ such that $s \in O$. Since O is open, we can choose $\varepsilon > 0$ such that $(s - \varepsilon, s] \subseteq O$. Also since $s = \sup X$, $[a, s - \varepsilon/2]$ is contained in the union of some finite subfamily $\mathscr{F}'$ of $\mathscr{F}$, $[a, s] = [a, s - \varepsilon/2] \cup (s - \varepsilon, s]$ is contained in the union $\mathscr{F}' \cup \{O\}$. This shows that $s \in X$.



- $s = b$
  Proof Let $O \in \mathscr{F}$ such that $s \in O$. Suppose that $s < b$, there is $\varepsilon > 0$ such that $(s - \varepsilon, s + \varepsilon) \subseteq O$.
  Since $s = \sup X$, $[a, s - \varepsilon/2]$ is contained in the union of some finite subfamily $\mathscr{F}'$ of $\mathscr{F}$, $[a, s + \varepsilon/2] = [a, s - \varepsilon/2] \cup (s - \varepsilon, s + \varepsilon/2]$ is contained in the union $\mathscr{F}' \cup \{O\}$. This shows that $s < s + \varepsilon/2 \in X$ and contradicting the fact that $s = \sup X$. Therefore $s = b$ and all of $[a, b]$ is contained in the union $\mathscr{F}' \cup \{O\}$.



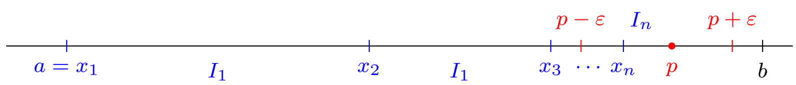
"Subdivision" Proof of Heine-Borel Theorem

Suppose that the Heine-Borel Theorem is false. Let $\mathscr{F}$ be an open cover of $[a, b]$ which does not contain a finite subcover.

- Set $I_1 = [a, b]$.
- Subdivide $[a, b]$ into 2 closed subintervals of equal length $[a, \frac{1}{2}(a + b)]$ and $[\frac{1}{2}(a + b), b]$. At least one of these must have the property that it is not contained in the union of any finite subfamily of $\mathscr{F}$. Select one of $[a, \frac{1}{2}(a + b)]$, $[\frac{1}{2}(a + b), b]$ which has this property and call it I_2.
- Now repeat the process, bisecting I_2 and selecting one half, called I_3, which is not contained in the union of any finite subfamily of $\mathscr{F}$.
- Continuing in this way, we obtain a nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots \quad \text{with the length of } I_n \text{ equals } |I_n| = \frac{b - a}{2^{n-1}} \quad \forall n = 1, 2, \dots$$

- For each $n \in \mathbb{N}$, let x_n be the left-hand end point of I_n. Since the sequence $\{x_n\}$ is monotonic increasing and bounded above, $p = \sup\{x_n \mid n \in \mathbb{N}\}$ exists. For each $n \in \mathbb{N}$, since $x_n \in I_n$ and $p = \lim_{k \to \infty} x_k$,



$$|x_n - p| \le \sum_{k=n}^{\infty} |x_k - x_{k+1}| \le \sum_{k=n}^{\infty} \frac{b - a}{2^k} = \frac{b - a}{2^{n-1}} = |I_n| \implies p \in I_n$$

and since $\lim_{n\to\infty} |I_n| = 0$, we have

$$\bigcap_{n=1}^{\infty} I_n = \{p\}.$$

- Since $p \in [a,b]$, there is an open set $O \in \mathscr{F}$, an $\varepsilon > 0$ and an $n \in \mathbb{N}$ such that $p \in O$, $(p-\varepsilon, p+\varepsilon) \cap [a,b] \subseteq O$ and $|I_n| < \varepsilon$. Also since $p \in I_n$, $I_n \subseteq (p-\varepsilon, p+\varepsilon) \cap [a,b] \subseteq O$, i.e. I_n is contained in a single element of $\mathscr{F}$, which is a contradiction to the choice of I_n.

Corollary A closed rectangular box $\prod_{k=1}^{n}[a_k, b_k]$ of $\mathbb{R}^n$ is compact.

Theorem If X is a compact topological space and if $f : X \to Y$ is an onto continuous function, then Y is compact.

Remark Compactness is a topological property, i.e. if X is compact and if X is homeomorphic to Y, then Y is compact.

Proof Let $\mathscr{F}$ be an open cover of Y. For each $O \in \mathscr{F}$, since $f : X \to Y$ is an onto continuous function, $f^{-1}(O)$ is an open subset of X and

$$\mathscr{G} = \{f^{-1}(O) \mid O \in \mathscr{F}\}$$

is an open cover of X, and the compactness of X implies that $\mathscr{G}$ contains a finite subcover, say

$$X = f^{-1}(O_1) \cup \cdots \cup f^{-1}(O_k).$$

Next since $f : X \to Y$ is an onto function, we have

$$f(f^{-1}(O_i)) = O_i \text{ for } 1 \leq i \leq k \text{ and } Y = \bigcup_{i=1}^{k} f(f^{-1}(O_i)) = \bigcup_{i=1}^{k} O_k.$$

So $\{O_i \mid 1 \leq i \leq k\}$ is a finite subcover of $\mathscr{F}$. This shows that Y is compact.

Theorem If X is a compact topological space and if C is a closed subset of X, then C is compact.

Proof Let $\mathscr{F}$ be a family of open subsets of X that covers C, i.e.

$$C \subseteq \bigcup \mathscr{F} = \bigcup_{O \in \mathscr{F}} O.$$

Since $(X \setminus C) \cup \mathscr{F}$ is an open cover of X and since X is compact, there exist $O_1, O_2, \ldots, O_k \in \mathscr{F}$ such that

$$X = \left(\bigcup_{i=1}^{k} O_i\right) \cup (X \setminus C) \implies C \subseteq \bigcup_{i=1}^{k} O_i$$

and $\{O_i \mid 1 \leq i \leq k\}$ is a finite subcover of $\mathscr{F}$. This shows that C is compact.

Definition A metric or distance function on a set X is a real-valued function $d : X \times X \to \mathbb{R}$ defined on the Cartesian product $X \times X$ such that for all x, y, $z \in X$:

(a) $d(x,y) \geq 0$ and equality holds if and only if $x = y$;

(b) $d(x,y) = d(y,x)$;

(c) $d(x,y) + d(y,z) \geq d(x,z)$.

A set X together with a metric d on it, usually denoted (X, d), is called a metric space (generated by $\mathscr{B} = \{B_r(p) \mid p \in X,\ 0 < r < 1\}$).

Definition A topological space X is called a Hausdorff space if two distinct points can always be surrounded by disjoint open sets, i.e.

$$\forall\ p \neq q \in X,\ \exists \text{ open subsets } U,\ V \text{ of } X \text{ such that } p \in U,\ q \in V \text{ and } U \cap V = \emptyset.$$

Theorem If A is a compact subset of a Hausdorff space X, and if $x \in X \setminus A$, then there exist disjoint neighborhoods of x and A. Therefore a compact subset of a Hausdorff space is closed.

Proof For each $z \in A$, since X is Hausdorff, let U_z and V_z be disjoint open subsets such that $x \in U_z$ and $z \in V_z$. Since

$$A \subseteq \bigcup_{z \in A} V_z,$$

$\mathscr{F} = \{V_z \mid z \in A\}$ is an open cover of A, and since A is compact there exist a finite subcover $\{V_{z_i} \mid z_i \in A,\ \text{for each } 1 \leq i \leq k\}$ of $\mathscr{F}$ such that

$$A \subseteq \bigcup_{i=1}^{k} V_{z_i}.$$

Let $V = \bigcup_{i=1}^{k} V_{z_i}$. Since $V_{z_i} \cap U_{z_i} = \emptyset$ and $x \in U_{z_i}$ for each $1 \leq i \leq k$, the sets $U = \bigcap_{i=1}^{k} U_{z_i}$ and V are disjoint open neighborhoods of x and A.

Theorem If X is a compact space, Y is a Hausdorff space and $f : X \to Y$ is a one-to-one, onto and continuous function, then $f : X \to Y$ is a homeomorphism.

Proof If C is a closed subset of X, since X is compact and $f : X \to Y$ is one-to-one, onto and continuous, C is compact in X and $\left(f^{-1}\right)^{-1}(C) = f(C)$ is compact and consequently closed in Y. So $f : X \to Y$ takes closed sets to closed sets which proves that $f^{-1} : Y \to X$ is continuous and $f : X \to Y$ is a homeomorphism.

(Bolzano-Weierstrass Property) An infinite set of points in a compact space must have a limit point, i.e. If S is an infinite subset of a compact space X, then $S' \cap X \neq \emptyset$.

Proof Let X be a compact space and let S be a subset of X which has no limit point, i.e.

$$S' \cap X = \emptyset.$$

For each $x \in X$, since $x \notin S'$, there is an open neighborhood $O(x)$ of x such that

$$O(x) \cap S \setminus \{x\} = \emptyset \implies O(x) \cap S = \begin{cases} \emptyset & \text{if } x \notin S \\ \{x\} & \text{if } x \in S \end{cases}$$

By the compactness of X, the open cover $\{O(x) \mid x \in X\}$ has a finite subcover. But each set $O(x)$ contains at most one point of S and therefore S must be a finite set.

Theorem A continuous real-valued function defined on a compact space is bounded and attains its bounds.

Proof If $f : X \to \mathbb{R}$ is continuous and if X is compact, then $f(X)$ is compact. Therefore $f(X)$ is bounded closed subset of $\mathbb{R}$ by a preceding theorem and there exist $x_1,\ x_2 \in X$ such that

$$f(x_1) = \sup(f(X)) \quad \text{and} \quad f(x_2) = \inf(f(X)).$$

(Lebesgue's Lemma) Let X be a compact metric space and let $\mathscr{F}$ be an open cover of X. Then there exists a real number $\delta > 0$ (called a Lebesgue number of $\mathscr{F}$) such that any subset of X of diameter less than δ is contained in some member of $\mathscr{F}$.

Definition Let A, B be subsets of the metric space (X, d). Then the diameter of A is defined by

$$\operatorname{diam}(A) = \sup_{x,\, y \in A} d(x, y)$$

and the distance $d(A, B)$ between A and B is defined by

$$d(A, B) = \inf_{x \in A,\, y \in B} d(x, y).$$

Proof If Lebesgue's Lemma is false, there exists a sequence $\{A_n \neq \emptyset \mid n \in \mathbb{N}\}$ of subsets of X such that

- $A_n \not\subseteq U$ for each $U \in \mathscr{F}$, $n \in \mathbb{N}$.
- $d(A_n) = \operatorname{diam}(A_n) \searrow 0$ (diameter of A_n deceases to 0).

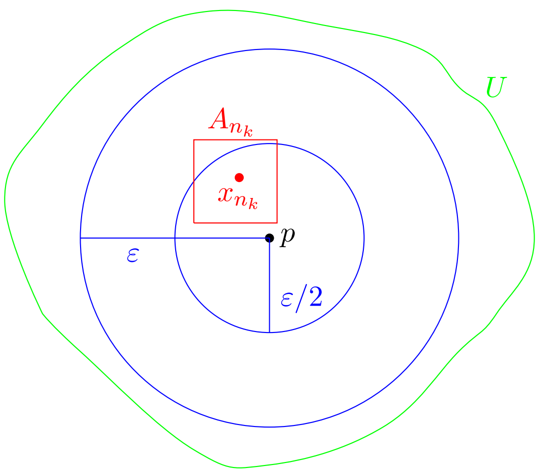
For each $n = 1, 2, \ldots$, choose a point $x_n \in A_n$. Then the sequence $\{x_n\}$ contains

- either finitely many distinct points (in which case some point repeats infinitely times)
- or infinitely many distinct points (in which case $\{x_n\}$ has a limit point since X is compact).

Denote the repeated point, or limit point, by p. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to p. Since $p \in X$ and $\mathscr{F}$ is an open cover of X, there is an open set $U \in \mathscr{F}$ containing p. Choose $\varepsilon > 0$ such that $B_\varepsilon(p) \subseteq U$, and choose an integer k large enough so that:

(a) $d(A_{n_k}) < \varepsilon/2 \implies d(x_{n_k}, x) < \varepsilon/2$ for all $x \in A_{n_k}$, and

(b) $d(x_{n_k}, p) < \varepsilon/2 \iff x_{n_k} \in B_{\varepsilon/2}(p)$.



Thus we have

$$d(x, p) \leq d(x, x_{n_k}) + d(x_{n_k}, p) < \varepsilon \quad \text{for all } x \in A_{n_k} \implies A_{n_k} \subseteq B_\varepsilon(p) \subseteq U.$$

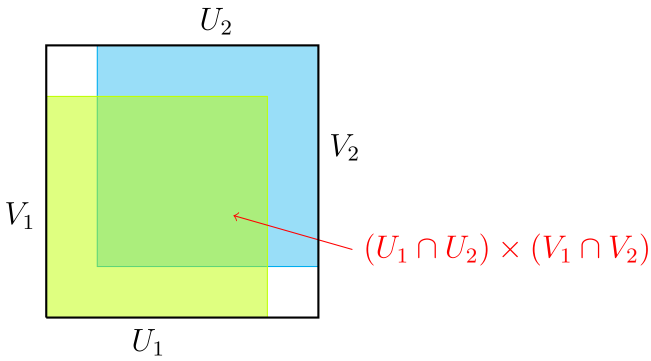
This contradicts our initial choice of the sequence $\{A_n\}$.

Product Spaces

Definition Let X and Y be topological spaces and let $\mathscr{B}$ denote the family of all subsets of $X \times Y$ of the form $U \times V$, where U is open in X and V is open in Y.

Since

- $\bigcup_{U \times V \in \mathscr{B}} U \times V = X \times Y$,
- $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathscr{B}$ for any $U_1 \times V_1$, $U_2 \times V_2 \in \mathscr{B}$,



$\mathscr{B}$ is a base for a topology on $X \times Y$. This topology is called the product topology, and the set $X \times Y$, when equipped with the product topology, is called a product space.

In general, if $X_1, X_2, \ldots, X_n$ are topological spaces, the product topology on $X_1 \times X_2 \times \cdots \times X_n$ is the topology generated by the base $\mathscr{B} = \{U_1 \times U_2 \times \cdots \times U_n \mid U_i \text{ is open in } X_i,\ 1 \leq i \leq n\}$.

The functions $\pi_i : X_1 \times \cdots \times X_i \times \cdots \times X_n \to X_i$ defined by $\pi_i(x_1, \cdots, x_i, \cdots, x_n) = x_i$ for $1 \leq i \leq n$, are called projections.

Theorem If $X \times Y$ has the product topology $\mathscr{T}$ then the projections are continuous functions and they take open sets to open sets. The product topology $\mathscr{T}$ is the smallest topology on $X \times Y$ for which both projections are continuous.

Proof Suppose U is an open subset of X and V is an open subset of Y, since $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$ are open in the product topology $\mathscr{T}$, π_1 and π_2 are continuous.

Since the product topology $\mathscr{T}$ is generated by the base $\mathscr{B} = \{U \times V \mid U \text{ is open in } X,\ V \text{ is open in } Y\}$, and since $\pi_1(U \times V) = U$ is open in X, $\pi_2(U \times V) = V$ is open in Y for each $U \times V \in \mathscr{B}$, projections π_1 and π_2 are open mappings.

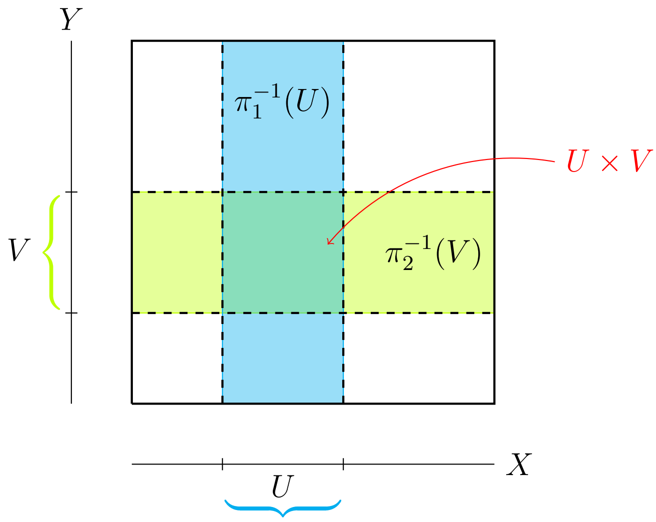
Let $\mathscr{T}'$ be a topology on $X \times Y$, so that both projections are continuous. So $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) \in \mathscr{T}'$ for any open subsets U of X and V of Y, and since

$$U \times V = (U \times Y) \cap (X \times V) = \pi_1^{-1}(U) \cap \pi_2^{-1}(V) \in \mathscr{T}' \implies \mathscr{T} \subseteq \mathscr{T}'.$$

This proves that the product topology $\mathscr{T}$ the smallest topology on $X \times Y$ for which both projections are continuous.

Theorem A function $f : Z \to X \times Y$ is continuous if and only if the two composite functions (coordinate functions) $\pi_1 \circ f : Z \to X$, $\pi_2 \circ f : Z \to Y$ are both continuous.

Proof ($\Longrightarrow$) If $f : Z \to X \times Y$ is continuous, then $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous, by the continuity of the projections π_1, π_2.



($\Longleftarrow$) If both $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous, then $f : Z \to X \times Y$ is continuous since for each basic open set $U \times V$ of $X \times Y$,

$$f^{-1}(U \times V) = (\pi_1 \circ f)^{-1}(U) \cap (\pi_2 \circ f)^{-1}(V) \quad \text{is open in } Z.$$

Theorem The product space $X \times Y$ is a Hausdorff space if and only if both X and Y are Hausdorff.

Proof ($\Longrightarrow$) Suppose that $X \times Y$ is Hausdorff. Given distinct points x_1, $x_2 \in X$, we choose a point $y \in Y$ and find disjoint basic open sets $U_1 \times V_1$, $U_2 \times V_2$ in $X \times Y$ such that $(x_1, y) \in U_1 \times V_1$ and $(x_2, y) \in U_2 \times V_2$.

Then U_1, U_2 are disjoint open neighborhoods of x_1 and x_2 in X. Therefore X is a Hausdorff space.

The argument for Y is similar.

($\Longleftarrow$) Suppose that X and Y are both Hausdorff spaces. Let (x_1, y_1) and (x_2, y_2) be distinct points of $X \times Y$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$ (or both).

If $x_1 \neq x_2$, since X is Hausdorff, there are disjoint open sets U_1, U_2 in X such that $x_1 \in U_1$ and $x_2 \in U_2$. Since $(x_1, y_1) \in U_1 \times Y$, $(x_2, y_2) \in U_2 \times Y$ and $(U_1 \times Y) \cap (U_2 \times Y) = \emptyset$, $X \times Y$ is a Hausdorff space.

The argument for $y_1 \neq y_2$ is similar.

Lemma Let X be a topological space and let $\mathscr{B}$ be a base for the topology of X. Then X is compact if and only if every open cover of X by members of $\mathscr{B}$ has a finite subcover.

Proof ($\Longrightarrow$) This is obvious since basis elements are open.

($\Longleftarrow$) Suppose that every open cover of X by members of $\mathscr{B}$ has a finite subcover, and let $\mathscr{F}$ be an arbitrary open cover of X.

Since $\mathscr{B}$ is a base for the topology of X, each member of $\mathscr{F}$ is a union of members of $\mathscr{B}$. Let $\mathscr{B}'$ denote the family of those members of $\mathscr{B}$ which are used in this process.

By construction we have

$$\bigcup_{B \in \mathscr{B}'} B = \bigcup_{U \in \mathscr{F}} U = X$$

so $\mathscr{B}'$ is an open cover of X (by members of $\mathscr{B}$) and must therefore contain a finite subcover.

For each basic open set in this finite subcover, we select a single member of $\mathscr{F}$ which contains it. This gives a finite subcover of $\mathscr{F}$ and shows that X is compact.

Theorem The product space $X \times Y$ is compact if and only if both X and Y are compact.

Proof ($\Longrightarrow$) If $X \times Y$ is compact, then both X and Y are compact since the projections $\pi_1 : X \times Y \to X$, $\pi_2 : X \times Y \to Y$ are onto and continuous functions.

($\Longleftarrow$) Suppose both X and Y are compact spaces and let $\mathscr{F}$ be an open cover of $X \times Y$ by basic open sets of the form $U \times V$, where U is open in X and V is open in Y.

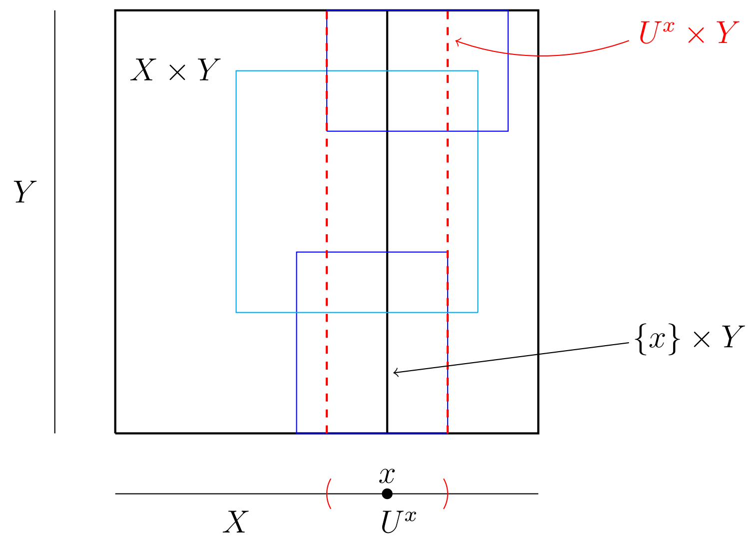
For each $x \in X$, consider the subset $\{x\} \times Y$ of $X \times Y$ with the induced topology. It is easy to check that

$$\pi_2|_{\{x\}\times Y} : \{x\} \times Y \to Y$$

is a homeomorphism. In other words $\{x\} \times Y$ is just a copy of Y in $X \times Y$ which lies 'over' the point x. So $\{x\} \times Y$ is compact and we can find a finite subfamily $\{U_i^x \times V_i^x \mid 1 \le i \le n_x\}$ of $\mathscr{F}$ whose union contains $\{x\} \times Y$. Since $x \in U_i^x$ for each $1 \le i \le n_x$, $U^x = \cap_{i=1}^{n_x} U_i^x \neq \emptyset$ and

$$U^x \times Y \subseteq \bigcup_{i=1}^{n_x} U_i^x \times V_i^x,$$

the union of these sets contains more than $\{x\}$, it actually contains all of $U^x \times Y$.



Since the family $\{U^x \mid x \in X\}$ is an open cover of X, we can select a finite subcover $\{U^{x_j} \mid 1 \le j \le s\}$ of X such that $X = \bigcup_{i=j}^{s} U^{x_j}$ and

$$X \times Y = \bigcup_{j=1}^{s} (U^{x_j} \times Y) \subseteq \bigcup_{j=1}^{s} \bigcup_{i=1}^{n_{x_j}} (U_i^{x_j} \times V_i^{x_j})$$

this implies that $X \times Y$ is compact since it can be covered by a finite subfamily $\{U_i^{x_j} \times V_i^{x_j} \mid 1 \le j \le s,\ 1 \le i \le n_{x_j}\}$ of $\mathscr{F}$.

Theorem A subset of $\mathbb{E}^n$ is compact if and only if it is closed and bounded.

Connectedness

Definition Let X be a topological space. A separation of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be connected if there does not exist a separation of X.

A space X is disconnected if there exists a separation U, V of X, or equivalently if there are subsets A, B of X such that

$$A \neq \emptyset,\ B \neq \emptyset,\ A \cup B = X,\ \bar{A} \cap B = A \cap \bar{B} = \emptyset.$$

Note that $A \cup B = X$, $\bar{A} \cap B = A \cap \bar{B} = \emptyset \implies \bar{A} \cup B = A \cup \bar{B} = X$ and the sets $A = X \setminus \bar{B}$ and $B = X \setminus \bar{A}$ are disjoint nonempty open (and closed) subsets of X.

Remark A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.

Proof ($\Longrightarrow$) If A is a nonempty proper subset of X (i.e. $A \subsetneq X$) which is both open and closed in X, then the sets $U = A$ and $V = X \setminus A$ constitute a separation of X, since

$$A,\ B \text{ are open (and closed), disjoint, nonempty, and } A \cup B = X.$$

($\Longleftarrow$) If U and V form a separation of X, then $U \neq \emptyset$, $U \neq X$, and $U = X \setminus V$ is both open and closed in X.

Theorem The real line $\mathbb{R}$ is a connected space.

Proof Suppose $\mathbb{R} = A \cup B$, where $A \neq \emptyset$, $B \neq \emptyset$ and $A \cap B = \emptyset$.

Choose points $a \in A$, $b \in B$ and (without loss of generality) suppose that $a < b$. Let

$$\text{Let } X = \{x \in A \mid x < b\} \text{ and let } s = \sup X \leq b.$$

Since $\mathbb{R} = A \cup B$, either $s \in A$, or $s \notin A$.

- If $s \in A$, then $s \lneq b$ since $b \in B$ and $A \cap B = \emptyset$. Also since $s = \sup X$, we have $(s, b) \subseteq B$ which implies that $s \in B' \subseteq \bar{B}$ and thus $A \cap \bar{B} \neq \emptyset$.
- If $s \notin A$, then $s \in B$ since $A \cap B = \emptyset$. Also since $s = \sup X$, we have $s \in X' \subseteq A' \subseteq \bar{A}$ and thus $\bar{A} \cap B \neq \emptyset$.

Remark If we replace the real line $\mathbb{R}$ by an interval I in the proof, we can show that any interval I is connected.

Theorem Let X be a nonempty subset of $\mathbb{R}$. Then X is connected if and only if X is an interval.

Proof ($\Longrightarrow$) If X is not an interval, then we can find points $a, b \in X$ and a point $p \notin X$ such that $a < p < b$.

$$\text{Let } A = \{x \in X \mid x < p\} \text{ and let } B = X \setminus A \implies A \neq \emptyset,\ B \neq \emptyset \text{ and } X = A \cup B.$$

However since $X = \bar{X} = \bar{A} \cup \bar{B}$, $\bar{A} \subseteq X$, $\bar{B} \subseteq X$, and since $p \notin X$, note that

- if $x \in \bar{A}$, then $x \lneq p \implies \bar{A} = A \implies \bar{A} \cap B = A \cap B = \emptyset$,
- if $x \in \bar{B}$, then $p \gneq x \implies \bar{B} = B \implies A \cap \bar{B} = A \cap B = \emptyset$.

This implies that X is not connected.

Theorem The following conditions on a space X are equivalent:

(a) X is connected.

(b) X and $\emptyset$ are the only subsets of X which are both open and closed.

(c) X cannot be expressed as the union of two disjoint nonempty open sets.

(d) There are no onto continuous function from X to a discrete space which contains more than one point.

Proof

$[(a) \iff (b)]$ done as in a preceding Remark above.

$[(b) \iff (c)]$ done as in the Definition.

$[(c) \Rightarrow (d)]$ Suppose (c) is satisfied, and let Y be a discrete space with more than one point and let $f : X \to Y$ be an onto continuous function.

Break up Y as a union $U \cup V$ of two disjoint nonempty open sets. Then $X = \left[f^{-1}(U)\right] \cup \left[f^{-1}(V)\right]$ which is the union of two disjoint nonempty open sets, contradicting (c).

$[(d) \Rightarrow (a)]$ Let X be a space which satisfies (d) and suppose X is not connected. There exist $A,\, B \subseteq X$ such that

$$A \neq \emptyset,\ B \neq \emptyset,\ A \cup B = X \text{ and } \bar{A} \cap B = A \cap \bar{B} = \emptyset.$$

Since $\bar{A}$, $\bar{B}$ are closed, and $A = X \setminus \bar{B}$, $B = X \setminus \bar{A}$, A, B are also open subsets of X. Define a function f from X to the subspace $\{-1,\, 1\}$ of $\mathbb{R}$ by

$$f(x) = \begin{cases} -1 & \text{if } x \in A \\ \ \ 1 & \text{if } x \in B. \end{cases}$$

Then f is continuous and onto, contradicting (d) for X.

Theorem If X is a connected space and if $f : X \to Y$ is an onto continuous function, then Y is connected.

Proof If A is a subset of Y which is both open and closed, then $f^{-1}(A)$ is both open and closed in X. Since X is connected, $f^{-1}(A)$ is either X or $\emptyset$, which implies that A is Y or $\emptyset$. This proves that Y is connected.

Remark Replacing Y by the subspace $f(X)$ of Y, the proof implies that if X is a connected space and if $f : X \to Y$ is a continuous function, then $f(X)$ is connected.

Corollary If $h : X \to Y$ is a homeomorphism, then X is connected if and only if Y is connected. In brief, connectedness is a topological property of a space.

Theorem Let X be a topological space and let Z be a subset of X. If Z is connected and if Z is dense in X (i.e. $\bar{Z} = X$), then X is connected.

Proof Let A be a nonempty subset of X which is both open and closed. Since Z is dense in X, so $X = \bar{Z} = Z \cup Z'$ and we **claim:**

$$U \cap Z \neq \emptyset \quad \text{for each nonempty open subset } U \text{ of } X.$$

Claim holds since

$$\text{if } U \cap Z = \emptyset \implies U \cap Z' = \emptyset \implies U = U \cap X = U \cap (Z \cup Z') = (U \cap Z) \cup (Z \cap Z') = \emptyset$$

Hence we have

$$A \cap Z \neq \emptyset.$$

Since A is both open and closed in X, $A \cap Z$ is both open and closed in Z, and since Z is connected and $A \cap Z \neq \emptyset$, we deduce that

$$A \cap Z = Z \implies Z \subseteq A \implies X = \bar{Z} \subseteq \bar{A} = A \subseteq X \implies A = X.$$

This implies that X is connected.

Remark Note that if Z is a connected subset of a topological space X, then Z is a connected subset of the subspace $\bar{Z}$. Replacing X by $\bar{Z}$, the proof implies that $\bar{Z}$ is connected. In fact, the proof implies the following Corollary holds.

Corollary If Z is a connected subset of a topological space X, and if $Z \subseteq Y \subseteq \bar{Z}$, then Y is connected. In particular, the closure $\bar{Z}$ of Z is connected.

Proof Since the closure of Z in Y is all of Y and by applying the preceding theorem to the pair $Z \subseteq Y$, one can show that Y is connected.

Theorem Let $\mathscr{F}$ be a family of subsets of a space X whose union is all of X. If each member of $\mathscr{F}$ is connected, and if no two members of $\mathscr{F}$ are separated from one another in X, then X is connected.

Proof Let A be a subset of X which is both open and closed. We shall show that A is either empty or equal to all of X.

For each $Z \in \mathscr{F}$, since Z is connected and $Z \cap A$ is both open and closed in Z, $Z \cap A = \emptyset$ or Z. Since $X = \bigcup_{Z \in \mathscr{F}} Z$, we must have

- either $Z \cap A = \emptyset$ for all $Z \in \mathscr{F} \implies A = \emptyset$,
- or there is a $Z_A \in \mathscr{F}$ such that $Z_A \cap A \neq \emptyset \implies Z_A \cap A = Z_A$ and $A \neq \emptyset \implies Z_A \subseteq A$.

Suppose that $A \neq \emptyset$ and $A \neq X$, since A, $X \setminus A$ are disjoint open and closed nonempty subsets of X, there exist Z_A, $Z_{X \setminus A} \in \mathscr{F}$ such that

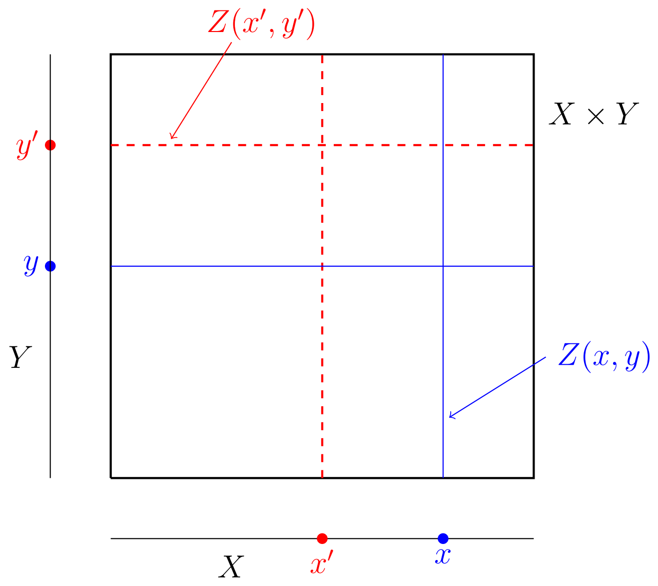
$$Z_A \cap A = Z_A \text{ and } Z_{X\setminus A} \cap (X \setminus A) = Z_{X\setminus A} \implies Z_A \subseteq A \text{ and } Z_{X\setminus A} \subseteq X \setminus A \implies Z_A \cap Z_{X\setminus A} = \emptyset$$

contradicting to the assumption that no two members of $\mathscr{F}$ are separated from one another in X, so X is connected.

Theorem If X and Y are connected spaces then the product space $X \times Y$ is connected

Proof For each $x \in X$ and $y \in Y$, let $Z(x,y) = (\{x\} \times Y) \cup (X \times \{y\})$ and let $\mathscr{F} = \{Z(x,y) \mid x \in X,\, y \in Y\}$. Since $\{x\} \times Y$ and $X \times \{y\}$ are connected and since $(\{x\} \times Y) \cap (X \times \{y\}) = \{(x,y)\} \neq \emptyset$, $(\{x\} \times Y) \cup (X \times \{y\})$ is connected.

Also since no two members of $\mathscr{F}$ are separated from one another in $X \times Y$, and since $X \times Y = \bigcup_{Z(x,y) \in \mathscr{F}} Z(x,y)$, the space $X \times Y$ is connected.



Definition An equivalence relation on a set X is a relation $\sim$ on X having the following three properties:

- (Reflexivity) $x \sim x$ for every $x \in X$.
- (Symmetry) If $x \sim y$, then $y \sim x$.
- (Transitivity) If $x \sim y$ and $y \sim z$, then $x \sim z$.

The equivalence class of an element $x \in X$, denoted by $[x]$, is the set defined by

$$[x] = \{y \in X \mid y \sim x\}.$$

It is easy to see that distinct equivalence classes are disjoint, i.e $[x] \cap [y]$ is either $\emptyset$ or all of $[x]$.

Definition Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subset of X containing both x and y. The equivalence class C_x of an element $x \in X$ is called a component (or "connected component") of X.

Remark

- Let C and D be connected subsets of X such that $C \cap D \neq \emptyset$. Then $C \cup D$ is connected.
- For each $x \in X$, the (connected) component C_x is the largest connected subset containing of x. Hence
$$C_x \cap C_y = \text{ either } \emptyset \text{ or } C_x = C_y \quad \forall x, y \in X.$$

Theorem Let X be a topological space and let C_x denote the component of X containing $x \in X$. Then

- For each $x \in X$, the component C_x is closed in X.
- For any $x, y \in X$, $C_x \cap C_y$ is either an empty set or all of $C_x = C_y$, i.e. distinct components are separated from one another in the space.

Proof Let C_x be a component of X containing x. Then C_x is connected, and so $\bar{C}_x$ is connected by a preceding Corollary. Since C_x is an equivalence class of X, we must have $C_x = \bar{C}_x$ and C_x is closed.

If C_x, C_y are components of X such that $C_x \cap C_y \neq \emptyset$ then, since $C_x \cup C_y$ is a connected subset of X containing both C_x and C_y, we must have $C_x \cup C_y = C_x$ and $C_x \cup C_y = C_y$ which implies that $C_x = C_y$. So, distinct components are separated from one another in the space.

Joining Points by Paths

Definition Given points x and y of the topological space X, a path in X from x to y is a continuous function $\gamma : [a, b] \to X$ of some closed interval in the real line into X, such that $\gamma(a) = x$ and $\gamma(b) = y$. A space X is said to be path-connected if every pair of points of X can be joined by a path in X.

Remark One can define an equivalence relation on X by setting $x \sim y$ if there is a path in X joining x to y. This is an equivalence relation since

- For each $x \in X$, the path γ defined by
$$\gamma(t) = x \quad t \in [a, b]$$
is a path in X joining x to x.
- if γ is a path in X joining x to y, then $-\gamma$ defined by
$$-\gamma(t) = \gamma(a + b - t) \quad t \in [a, b]$$
is a (reversed) path in X joining y to x.
- if α, β are paths in X joining x to y and y to z respectively, then γ defined by
$$\gamma(t) = \begin{cases} \alpha(2t - a) & \text{if} \quad a \leq t \leq \dfrac{a+b}{2} \\ \beta(2t - b) & \text{if} \quad \dfrac{a+b}{2} \leq t \leq b \end{cases}$$
is a path in X joining x to z.

The equivalence classes are called the components (or "path-connected components") of X.

Theorem If X is a path-connected space, then X is connected.

Proof Suppose $X = A \cup B$ is a separation of X.

Let $\gamma : [a, b] \to X$ be any path in X. Since γ is continuous and $[a, b]$ is connected, the set $\gamma([a, b])$ is connected and, since

$$\begin{aligned} &\gamma([a, b]) = \gamma([a, b]) \cap X = \gamma([a, b]) \cap (A \cup B) = (\gamma([a, b]) \cap A) \cup (\gamma([a, b]) \cap B), \\ \implies \quad &\text{either } (\gamma([a, b]) \cap A) = \emptyset \text{ or } (\gamma([a, b]) \cap B) = \emptyset. \end{aligned}$$

i.e. $\gamma([a, b])$ lies entirely in either A or B.

This implies that there does not exist any path in X joining a point in A to a point in B, contrary to the assumption that X is path connected.

Theorem If X is a connected open subset of the Euclidean space $\mathbb{E}^n$, then X is path-connected.

Proof Given $x \in X$, let $U(x)$ be the collection of points of X defined by

$$U(x) = \{y \in X \mid y \text{ can be joined to } x \text{ by a path in } X\}.$$

Then $U(x) \neq \emptyset$ and $U(x)$ is a path connected subset (component) of X.

Claim For each $x \in X$, $U(x)$ is open in X.

Proof of Claim Let $y \in U(x)$, since X is open in $\mathbb{E}^n$, there exists a ball $B_r(y)$ such that $B_r(y) \subseteq X$. If $z \in B_r(y)$, since z can be joined to x by a path in X and $U(x)$ is a path-connected component of X, we must have $z \in U(x)$ and $B_r(y) \subseteq X$. This implies that $U(x)$ is open in X.

Claim For each $x \in X$, $U(x)$ is closed in X.

Proof of Claim Since

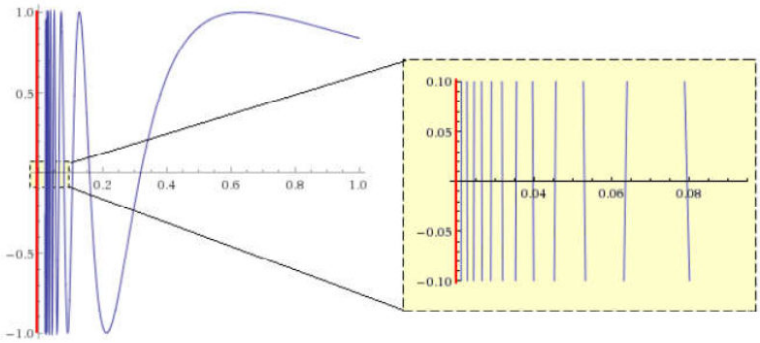
$$X \setminus U(x) = \bigcup_{y \in X \setminus U(x)} U(y) = \text{ union of open subset } U(y) \text{ of } X,$$

$X \setminus U(x)$ is open in X and thus $U(x)$ is closed in X.

Since X is connected and $U(x) \neq \emptyset$, we must have $U(x) = X$ which implies that X is path-connected.

The converse is not true: the topologist's sine curve is connected but not path-connected.

Example Let $Z = \{(x, \sin(1/x)) \mid 0 < x \leq 1\}$, $Y = \{0\} \times [-1, 1]$ and $X = Y \cup Z \subset \mathbb{R}^2$ be the topologist's sine curve. Since Z is path-connected, it is connected and $\bar{Z} = X$ is connected.



To see why it's not path-connected, suppose $f : [0, 1] \to X$ is continuous and $f(0) = (0, 0)$, $f(1) = (1, \sin 1)$.

Let π_x, $\pi_y : X \to \mathbb{R}$ be projection maps to the $x-$ and y-coordinates respectively. Since $\pi_x \circ f$ is continuous on $[0, 1]$, $\pi_x \circ f(0) = 0 < 1 = \pi_x \circ f(1)$, so its image $\mathrm{im}\,(\pi_x \circ f)$ is the whole $[0, 1]$ by the intermediate value theorem and hence, $Z \subset \mathrm{im}\,(f)$. Pick points $t_0, t_1, t_2, \ldots \in [0, 1]$ such that $f(t_n) = \left((2n\pi + \frac{\pi}{2})^{-1}, 1\right)$.

Since $[0, 1]$ is compact, $\pi_y \circ f$ is uniformly continuous. So for $\varepsilon = 1 > 0$, there exists $\delta > 0$ such that whenever $t, u \in [0, 1]$ satisfy $|t - u| < \delta$, we have $|\pi_y(f(t)) - \pi_y(f(u))| < 1$.

Since $\{t_k\}_{k=0}^{\infty}$ is an infinite sequence in the compact set $[0, 1]$, it contains a convergent subsequence, which is again denoted by $\{t_k\}_{k=0}^{\infty}$, and with the $\delta > 0$, there is an $m \in \mathbb{N}$ such that if $n > m$, then $|t_m - t_n| < \delta$.

Since $Z \subset \mathrm{im}\,(f)$, $f(t_m) = \left((2m\pi + \frac{\pi}{2})^{-1}, 1\right)$ and $f(t_n) = \left((2n\pi + \frac{\pi}{2})^{-1}, 1\right)$, there's a point u between t_m and t_n such that $f(u) = \left((2m\pi + \frac{3\pi}{2})^{-1}, -1\right)$. Then t_m and u satisfy $|t_m - u| < |t_m - t_n| < \delta$, but $|\pi_y(f(t_m)) - \pi_y(f(u))| = |1 - (-1)| = 2 > 1$ which is a contradiction.